

# On the Blow-up for a Discrete Boltzmann Equation in the Plane

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**Abstract.** We study the possibility of finite-time blow-up for a two dimensional Broadwell model. In a set of rescaled variables, we prove that no self-similar blow-up solution exists, and derive some a priori bounds on the blow-up rate. In the final section, a possible blow-up scenario is discussed.

## 1 - Introduction

Consider the simplified model of a gas whose particles can have only finitely many speeds, say  $c_1, \dots, c_N \in \mathbb{R}^n$ . Call  $u_i = u_i(t, x)$  the density of particles with speed  $c_i$ . The evolution of these densities can then be described by a semilinear system of the form

$$\partial_t u_i + c_i \cdot \nabla u_i = \sum_{j,k} a_{ijk} u_j u_k \quad i = 1, \dots, N. \quad (1.1)$$

Here the coefficient  $a_{ijk}$  measures the rate at which new  $i$ -particles are created, as a result of collisions between  $j$ - and  $k$ -particles. In a realistic model, these coefficients must satisfy a set of identities, accounting for the conservation of mass, momentum and energy.

Given a continuous, bounded initial data

$$u_i(0, x) = \bar{u}_i(x), \quad (1.2)$$

on a small time interval  $t \in [0, T]$  a solution of the Cauchy problem can be constructed by the method of characteristics. Indeed, since the system is semilinear, this solution is obtained as the

fixed point of the integral transformation

$$u_i(t, x) = \bar{u}_i(x - c_i t) + \int_0^t \sum_{j,k} a_{ijk} u_j u_k(s, x - c_i(t-s)) ds. \quad (1.3)$$

For sufficiently small time intervals, the existence of a unique fixed point follows from the contraction mapping principle, without any assumption on the constants  $a_{ijk}$ .

If the initial data is suitably small, the solution remains uniformly bounded for all times [3]. For large initial data, on the other hand, the global existence and stability of solutions is known only in the one-dimensional case [2, 6, 10]. Since the right hand side has quadratic growth, it might happen that the solution blows up in finite time. Examples where the  $\mathbf{L}^\infty$  norm of the solution becomes arbitrarily large as  $t \rightarrow \infty$  are easy to construct [7]. In the present paper we focus on the two-dimensional Broadwell model and examine the possibility that blow-up actually occurs in finite time.

Since the equations (1.1) admit a natural symmetry group, one can perform an asymptotic rescaling of variables and ask whether there is a blow-up solution which, in the rescaled variables, converges to a steady state. This technique has been widely used to study blow-up singularities of reaction-diffusion equations with superlinear forcing terms [4, 5]. See also [9] for an example of self-similar blow-up for hyperbolic conservation laws. Our results show, however, that for the two-dimensional Broadwell model no such self-similar blow-up solution exists.

If blow-up occurs at a time  $T$ , our results imply that for times  $t \rightarrow T-$  one has

$$\|u(t)\|_{\mathbf{L}^\infty} > \frac{1}{5} \frac{\ln |\ln(T-t)|}{T-t}. \quad (1.4)$$

This means that the blow-up rate must be different from the natural growth rate  $\|u(t)\|_{\mathbf{L}^\infty} = \mathcal{O}(1) \cdot (T-t)^{-1}$  which would be obtained in case of a quadratic equation  $\dot{u} = C u^2$ .

In the final section of this paper we discuss a possible scenario for blow-up. The analysis highlights how carefully chosen should be the initial data, if blow-up is ever to happen. This suggests that finite time blow-up is a highly non-generic phenomenon, something one would not expect to encounter in numerical simulations.

## 2 - Coordinate rescaling

In the following, we say that  $P^* = (t^*, x^*)$  is a **blow-up point** if

$$\limsup_{x \rightarrow x^*, t \rightarrow t^*-} u_i(t, x) = \infty$$

for some  $i \in \{1, \dots, N\}$ . Define the constant

$$C \doteq \max_i |c_i|.$$

We say that  $(t^*, x^*)$  is a **primary blow-up point** if it is a blow-up point and the backward cone

$$\Gamma \doteq \{(t, x); |x - x^*| < 2C(t^* - t)\}$$

does not contain any other blow-up point.

**Lemma 1.** *Let  $u = u(t, x)$  be a solution of the Cauchy problem (1.1)-(1.2) with continuous initial data. If no primary blow-up point exist, then  $u$  is continuous on the whole domain  $[0, \infty[ \times I\!\!R^n$ .*

**Proof.** If  $u$  is not continuous, it must be unbounded in the neighborhood of some point. Hence some blow-up point exists. Call  $\mathcal{B}$  the set of such blow-up points. Define the function

$$\varphi(x) \doteq \inf_{(\tau, \xi) \in \mathcal{B}} \{\tau + C|x - \xi|\}.$$

By Ekeland's variational principle (see [1], p.254), there exists a point  $x^*$  such that

$$\varphi(x) \geq \varphi(x^*) - \frac{C}{2}|x - x^*|$$

for all  $x \in I\!\!R^2$ . Then  $P^* \doteq (\varphi(x^*), x^*)$  is a primary blow-up point.  $\square$

Let now  $(t^*, x^*)$  be a primary blow-up point. One way to study the local asymptotic behavior of  $u$  is to rewrite the system in terms of the rescaled variables  $w_i = w_i(\tau, \eta)$ , defined by

$$\begin{cases} \tau = -\ln(t^* - t), \\ \eta = e^\tau x = \frac{x - x^*}{t^* - t}, \\ w_i = e^{-\tau} u_i = (t^* - t)u_i. \end{cases} \quad (2.1)$$

The corresponding system of evolution equations is

$$\partial_\tau w_i + (c_i + \eta) \cdot \nabla_\eta w_i = -w_i + \sum_{j,k} a_{ijk} w_j w_k \quad i = 1, \dots, n. \quad (2.2)$$

Any nontrivial stationary or periodic solution  $w$  of (2.2) would yield a solution  $u$  of (1.1) which blows up at  $(t^*, x^*)$ . On the other hand, the non-existence of such solutions for (2.2) would suggest that finite time blow-up for (1.1) is unlikely.

### 3 - The two-dimensional Broadwell model

Consider a system on  $I\!\!R^2$  consisting of 4 types particles (fig. 1), with speeds

$$c_1 = (1, 1), \quad c_2 = (1, -1), \quad c_3 = (-1, -1), \quad c_4 = (-1, 1).$$

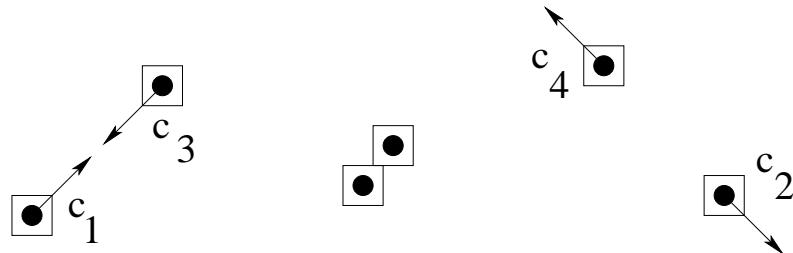


figure 1

The evolution equations are

$$\begin{cases} \partial_t u_1 + c_1 \cdot \nabla u_1 = u_2 u_4 - u_1 u_3, \\ \partial_t u_3 + c_3 \cdot \nabla u_3 = u_2 u_4 - u_1 u_3, \\ \partial_t u_2 + c_2 \cdot \nabla u_2 = u_1 u_3 - u_2 u_4, \\ \partial_t u_4 + c_4 \cdot \nabla u_4 = u_1 u_3 - u_2 u_4. \end{cases} \quad (3.1)$$

After renaming variables, the corresponding rescaled system (2.2) takes the form

$$\begin{cases} \partial_t w_1 + (x+1)\partial_x w_1 + (y+1)\partial_y w_1 = w_2 w_4 - w_1 w_3 - w_1, \\ \partial_t w_3 + (x-1)\partial_x w_3 + (y-1)\partial_y w_3 = w_2 w_4 - w_1 w_3 - w_3, \\ \partial_t w_2 + (x+1)\partial_x w_2 + (y-1)\partial_y w_2 = w_1 w_3 - w_2 w_4 - w_2, \\ \partial_t w_4 + (x-1)\partial_x w_4 + (y+1)\partial_y w_4 = w_1 w_3 - w_2 w_4 - w_4. \end{cases} \quad (3.2)$$

Our first result rules out the possibility of asymptotically self-similar blow-up solutions. A sharper estimate will be proved later.

**Theorem 1.** *The system (3.2) admits no nontrivial positive bounded solution which is constant or periodic in time.*

**Proof.** Assume

$$0 \leq w_i(t, x, y) \leq \kappa \quad (3.3)$$

for all  $t, x, y$ ,  $i = 1, 2$ . Choose  $\varepsilon \doteq e^{-2\kappa}/2$ , so that

$$\varepsilon < \frac{1}{\kappa}, \quad \varepsilon e^{2\kappa x} \leq \frac{1}{2} \quad x \in [-1, 1].$$

Define

$$\begin{aligned} Q_{14}(t, y) &\doteq \int_{-1}^1 \left[ (1 - \varepsilon e^{2\kappa x}) w_1(t, x, y) + (1 - \varepsilon e^{-2\kappa x}) w_4(t, x, y) \right] dx, \\ Q_{14}(t) &\doteq \sup_{|y| \leq 1} Q_{14}(t, y), \end{aligned}$$

Restricted to any horizontal moving line  $y = y(t)$  such that  $\dot{y} = y + 1$  (fig. 2), the equations (3.2) become

$$\begin{aligned} \partial_t w_1 + (x+1)\partial_x w_1 &= w_2 w_4 - w_1 w_3 - w_1, \\ \partial_t w_4 + (x-1)\partial_x w_4 &= w_1 w_3 - w_2 w_4 - w_4. \end{aligned}$$

A direct computation now yields

$$\begin{aligned} \frac{d}{dt} Q_{14}(t, y(t)) &\leq -2\varepsilon\kappa \int_{-1}^1 \left[ e^{2\kappa x}(1+x)w_1 + e^{-2\kappa x}(1-x)w_4 \right] dx + \int_{-1}^1 (\varepsilon e^{2\kappa x} - \varepsilon e^{-2\kappa x})(w_1 w_3 - w_2 w_4) dx \\ &\leq -\varepsilon\kappa \int_{-1}^1 \left[ e^{2\kappa x}(1+x)w_1 + e^{-2\kappa x}(1-x)w_4 \right] dx - \varepsilon\kappa \int_{-1}^0 e^{-2\kappa x}(1-x)w_4 dx \\ &\quad - \varepsilon\kappa \int_0^1 e^{2\kappa x}(1+x)w_1 dx + \int_{-1}^0 \varepsilon\kappa e^{-2\kappa x}w_4 dx + \int_0^1 \varepsilon\kappa e^{2\kappa x}w_1 dx \\ &\leq -\varepsilon\kappa \int_{-1}^1 \left[ e^{2\kappa x}(1+x)w_1 + e^{-2\kappa x}(1-x)w_4 \right] dx. \end{aligned}$$

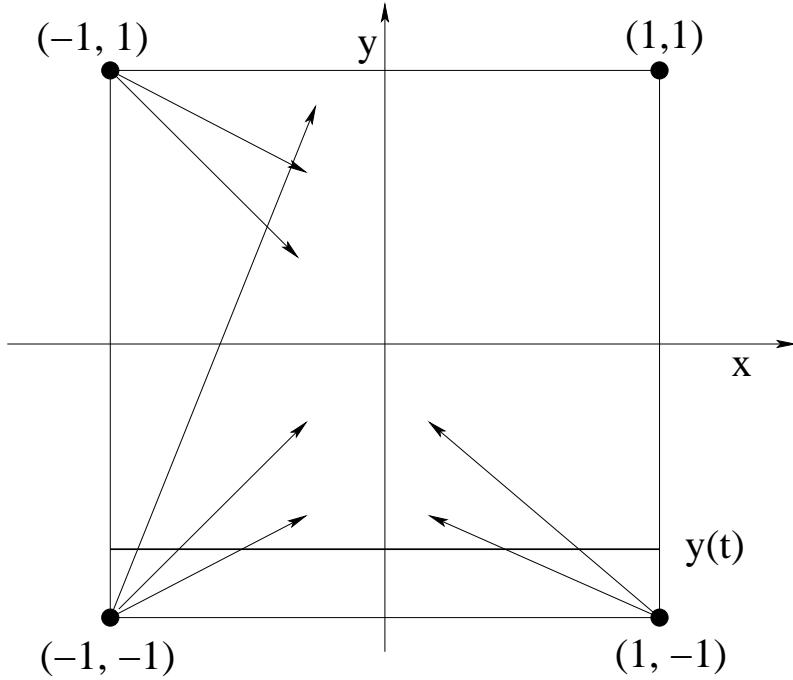


figure 2

Call

$$I(t, y) \doteq \int_{-1}^1 w_1(t, x, y) dx.$$

The definition of  $\varepsilon$  and the bound (3.3) on  $w_1$  imply

$$\begin{aligned} \int_{-1}^1 e^{2\kappa x} (1+x) w_1 dx &\geq 2\varepsilon \int_{-1}^1 (1+x) w_1 dx \\ &\geq 2\varepsilon \int_{-1}^{-1+I/\kappa} (1+x) \kappa dx \\ &= \varepsilon I^2 / \kappa \end{aligned}$$

From this, and a similar estimate for  $w_4$ , we obtain

$$\int_{-1}^1 [e^{2\kappa x} (1+x) w_1 + e^{-2\kappa x} (1-x) w_4] dx \geq \frac{\varepsilon}{\kappa} \left( \int_{-1}^1 w_1 dx \right)^2 + \frac{\varepsilon}{\kappa} \left( \int_{-1}^1 w_4 dx \right)^2 \geq \frac{\varepsilon}{\kappa} \frac{Q_{14}^2}{2}.$$

Since  $\varepsilon < \kappa^{-1}$ , this yields

$$\frac{d}{dt} Q_{14}(t, y(t)) \leq -\frac{\varepsilon^2}{2} Q_{14}^2(t, y(t)). \quad (3.4)$$

Observing that the Cauchy problem

$$\dot{z} = -\frac{\varepsilon^2}{2} z^2, \quad z(0) = 4\kappa$$

has the solution

$$z(t) = \left( \frac{1}{4\kappa} + \frac{\varepsilon^2}{2} t \right)^{-1},$$

by a comparison argument from (3.4) we deduce

$$Q_{14}(t) \leq \left( \frac{1}{4\kappa} + \frac{\varepsilon^2}{2} t \right)^{-1}.$$

Since

$$\int_{-1}^1 \int_{-1}^1 w_1(t, x, y) dx dy \leq 4Q_{14}(t),$$

and since a similar estimate can be performed for all components  $w_i$ , we conclude

$$\int_{-1}^1 \int_{-1}^1 w_i(t, x, y) dx dy \leq 4 \left( \frac{1}{4\kappa} + \frac{e^{-4\kappa}}{8} t \right)^{-1}. \quad (3.5)$$

The right hand side of (3.5) approaches zero as  $t \rightarrow \infty$ . Therefore, nontrivial constant or time-periodic  $\mathbf{L}^\infty$  solutions of (3.2) cannot exist.  $\square$

## 4 - Refined blow-up estimates

If  $(t^*, x^*)$  is a blow-up point, our analysis has shown that in the rescaled coordinates  $\tau, \xi$  the corresponding functions  $w_i$  must become unbounded as  $\tau \rightarrow \infty$ . In this section we refine the previous result, establishing a lower bound for the rate at which such explosion takes place.

**Theorem 2.** *Let  $u$  be a continuous solution of the Broadwell system (2.1). Fix any point  $(t^*, x^*)$  and consider the corresponding rescaled variables  $\tau, \xi, w_i$ . If*

$$\max_{|\xi_1|, |\xi_2| \leq 1} w_i(\tau, \xi_1, \xi_2) \leq \theta \ln \tau \quad i = 1, 2, 3, 4, \quad (4.1)$$

for some  $\theta < 1/4$  and all  $\tau$  sufficiently large, then

$$\lim_{\tau \rightarrow \infty} w_i(\tau, \xi) = 0 \quad i = 1, 2, 3, 4, \quad (4.2)$$

uniformly for  $\xi \in \mathbb{R}^2$  in compact sets. Therefore  $(t^*, x^*)$  is not a blow up point.

Since  $w_i = (t^* - t)u_i$  and  $\tau \doteq |\ln(t^* - t)|$ , the above implies

**Corollary 1.** *If  $(t^*, x^*)$  is a primary blow-up point, then*

$$\limsup_{x \rightarrow x^*, t \rightarrow t^*^-} |u(t, x)| \cdot \frac{t^* - t}{\ln |\ln(t^* - t)|} \geq \frac{1}{4}.$$

### Proof of Theorem 2.

Let  $w_i = w_i(t, x, y)$  provide a solution to the system (3.2), with

$$0 \leq w_i(t, x, y) \leq \theta \ln t \doteq k(t) \quad (4.3)$$

for all  $t \geq t_0$  and  $x, y \in [-1, 1]$ . The proof will be given in two steps. First we show that the  $\mathbf{L}^1$  norm of the components  $w_i$  approaches zero as  $t \rightarrow \infty$ . Then we refine the estimates, and prove that also the  $\mathbf{L}^\infty$  norm asymptotically vanishes.

STEP 1: Integral estimates. Consider the function

$$Q_{14}(t, y) \doteq \int_{-1}^1 \left[ \left(1 - \frac{e^{2k(t)(x-1)}}{2}\right) w_1(t, x, y) + \left(1 - \frac{e^{-2k(t)(x+1)}}{2}\right) w_4(t, x, y) \right] dx$$

with  $k(t)$  as in (4.3). As in the proof of Theorem 1, let  $t \mapsto y(t)$  be a solution to  $\dot{y} = y + 1$ . Then

$$\begin{aligned} \frac{d}{dt} Q_{14}(t, y(t)) &= \int_{-1}^1 \left[ -(x-1)k' e^{2k(t)(x-1)} w_1 + (x+1)k' e^{-2k(t)(x+1)} w_4 \right] dx \\ &\quad + \int_{-1}^1 \left(1 - \frac{e^{2k(t)(x-1)}}{2}\right) \left[ -(x+1)w_{1x} + w_2 w_4 - w_1 w_3 - w_1 \right] dx \\ &\quad + \int_{-1}^1 \left(1 - \frac{e^{-2k(t)(x+1)}}{2}\right) \left[ -(x-1)w_{4x} + w_1 w_3 - w_2 w_4 - w_4 \right] dx \end{aligned}$$

To estimate the right hand side, we notice that

$$\begin{aligned} A &\doteq \int_{-1}^1 \left(1 - \frac{e^{2k(t)(x-1)}}{2}\right) [(1+x)w_{1x} + w_1] dx \geq k(t) \int_{-1}^1 (x+1)e^{2k(x-1)} w_1 dx \\ B &\doteq \int_{-1}^1 \left(1 - \frac{e^{-2k(t)(x+1)}}{2}\right) [(x-1)w_{4x} + w_4] dx \geq k(t) \int_{-1}^1 (1-x)e^{-2k(x+1)} w_4 dx \\ C &\doteq \int_{-1}^1 (w_1 w_3 - w_2 w_4) \left(\frac{e^{2k(x-1)}}{2} - \frac{e^{-2k(x+1)}}{2}\right) dx \\ &\leq k(t) \int_0^1 \frac{e^{2k(x-1)}}{2} w_1 dx + k(t) \int_{-1}^0 \frac{e^{-2k(x+1)}}{2} w_4 dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{dt} Q_{14}(t, y(t)) &= \int_{-1}^1 \left[ -(x-1)k' e^{2k(t)(x-1)} w_1 + (x+1)k' e^{-2k(t)(x+1)} w_4 \right] dx - A - B + C \\ &\leq k'(t) \int_{-1}^1 [(1-x)e^{2k(t)(x-1)} w_1 + (1+x)e^{-2k(t)(x+1)} w_4] dx \\ &\quad - \frac{k(t)}{2} \int_{-1}^1 [(1+x)e^{2k(t)(x-1)} w_1 + (1-x)e^{-2k(t)(x+1)} w_4] dx. \end{aligned}$$

If  $k(t) \geq 1/2$ , we claim that the following two inequalities hold:

$$\begin{aligned} (1-x)e^{2k(t)(x-1)} &\leq 1 - e^{2k(t)(x-1)}, \\ (1+x)e^{-2k(t)(x+1)} &\leq 1 - e^{-2k(t)(x+1)}. \end{aligned} \quad (4.4)$$

To prove the first inequality we need to show that

$$h_k(s) \doteq 1 - e^{2ks} + se^{2ks} \geq 0 \quad \text{for all } s \in [-2, 0].$$

This is clear because  $h_k(0) = 0$  and

$$h'_k(s) = e^{2ks}(1 - 2k + 2ks) \leq 0 \quad s \in [-2, 0]$$

if  $k \geq 1/2$ . Hence  $h_k(s)$  is positive for  $s \in [-2, 0]$ , as claimed. The second inequality in (4.4) is proved similarly.

When  $t \geq t_0 \doteq e^{1/(2\theta)}$  one has  $k(t) \geq \frac{1}{2}$  and hence

$$\frac{d}{dt} Q_{14}(t, y(t)) \leq k'(t)Q_{14} - \frac{k(t)}{2} \int_{-1}^1 [(1+x)e^{2k(t)(x-1)}w_1 + (1-x)e^{-2k(t)(x+1)}w_4]dx.$$

Setting  $I = \int_{-1}^1 w_1 dx$ , we obtain

$$\int_{-1}^1 (1+x)e^{2k(t)(x-1)}w_1 dx \geq \int_{-1}^{-1+I/k(t)} (1+x)e^{-4k(t)}k(t) dx = e^{-4k(t)} \frac{I^2}{2k(t)}.$$

Using the above, and a similar estimate for the integral of  $w_4$ , we obtain

$$\begin{aligned} & \frac{k(t)}{2} \int_{-1}^1 [(1+x)e^{2k(t)(x-1)}w_1 + (1-x)e^{-2k(t)(x+1)}w_4]dx \\ & \geq \frac{e^{-4k(t)}}{4} \left[ \left( \int_{-1}^1 w_1 dx \right)^2 + \left( \int_{-1}^1 w_4 dx \right)^2 \right] \\ & \geq \frac{e^{-4k(t)}}{8} Q_{14}^2. \end{aligned} \tag{4.5}$$

Calling

$$Q_{14}(t) = \max_{|y| \leq 1} Q_{14}(t, y),$$

from (4.5) we deduce

$$\frac{d}{dt} Q_{14}(t) \leq k'(t)Q_{14}(t) - \frac{e^{-4k(t)}}{8} Q_{14}(t)^2.$$

Recalling that  $k(t) = \theta \ln t$  for some  $0 < \theta < 1/4$ , the previous differential inequality can be written as

$$\frac{d}{dt} Q_{14} \leq \frac{\theta}{t} Q_{14} - \frac{1}{8t^{4\theta}} Q_{14}^2. \tag{4.6}$$

Notice that  $Q_{14}(t_0, y(t_0)) \leq 2k(t_0)$ , and define the constant

$$A_0 \doteq \max \{2k(t_0)t_0^{1-4\theta}, 8(1-3\theta)\}.$$

Then the function

$$z(t) \doteq A_0 t^{4\theta-1}$$

satisfies

$$\frac{d}{dt} z(t) \geq \frac{\theta}{t} z - \frac{1}{t^{4\theta}} z^2 \quad z(t_0) \geq Q_{14}(t_0, y(t_0)). \tag{4.7}$$

Comparing (4.6) with (4.7) we conclude

$$Q_{14}(t) \leq z(t) \quad t \geq t_0. \quad (4.8)$$

This implies the estimate

$$\int_{-1}^1 w_i(t, x, y_0) dx \leq 2Q_{14}(t) \leq 2A_0 t^{4\theta-1}$$

for  $t \geq t_0$ ,  $i \in \{1, 4\}$  and any  $y_0 \in [-1, 1]$ . An entirely similar argument applied to  $Q_{12}, Q_{23}, \dots$  yields the estimates

$$\int_{-1}^1 w_i(t, x, y_0) dx \leq 2A_0 t^{4\theta-1}, \quad \int_{-1}^1 w_i(t, x_0, y) dy \leq 2A_0 t^{4\theta-1}. \quad (4.9)$$

for  $i = 1, 2, 3, 4$ ,  $x_0, y_0 \in [-1, 1]$  and  $t \geq t_0$ .

STEP 2: Pointwise estimates. Using the integral bounds (4.9), we now seek a uniform bound of the form

$$w_i(t, x, y) \leq C_0 \quad (4.10)$$

for some constant  $C_0$  and all  $x, y \in [-1, 1]$ ,  $t > 0$ .

To prove (4.10), let  $t \mapsto x(t)$ ,  $t \mapsto y(t) \in [-1, 1]$  be solutions of

$$\dot{x} = x + 1, \quad \dot{y} = y + 1.$$

Call

$$A(t) \doteq \int_{x(t)}^1 (w_1 + w_4)(t, x, y(t)) dx.$$

From our previous estimates (4.9) it trivially follows

$$A(t) \leq 4A_0 t^{4\theta-1}. \quad (4.11)$$

The time derivative of  $A(t)$  is computed as

$$\begin{aligned} \frac{dA}{dt} &= - (x(t) + 1)(w_1 + w_4)(t, x(t), y(t)) + \int_{x(t)}^1 [\partial_t w_1 + (y(t) + 1)\partial_y w_1 + \partial_t w_4 + (y(t) + 1)\partial_y w_4] dx \\ &= - (x(t) + 1)(w_1 + w_4)(t, x(t), y(t)) - \int_{x(t)}^1 [w_1 + (x + 1)\partial_x w_1 + w_4 + (x - 1)\partial_x w_4] dx \\ &= - (x(t) + 1)(w_1 + w_4)(t, x(t), y(t)) - \int_{x(t)}^1 (w_1 + w_4) dx \\ &\quad - 2w_1(t, 1, y(t)) + (x(t) + 1)w_1(t, x(t), y(t)) + \int_{x(t)}^1 w_1 dx \\ &\quad - 2w_4(t, 1, y(t)) + (x(t) - 1)w_4(t, x(t), y(t)) + \int_{x(t)}^1 w_4 dx \\ &\leq [x(t) - 1 - (x(t) + 1)]w_4(t, x(t), y(t)) = -2w_4(t, x(t), y(t)). \end{aligned}$$

This implies

$$w_4(t, x(t), y(t)) \leq -\frac{1}{2} \frac{dA}{dt}. \quad (4.12)$$

The total derivative of  $w_1$  along a characteristic line is now given by

$$\begin{aligned} \frac{d}{dt} w_1(t, x(t), y(t)) &= w_2 w_4 - w_1 w_3 - w_1 \leq w_2 w_4 - w_1 \leq \frac{1}{2} w_2 \left( \frac{-dA}{dt} \right) - w_1 \\ &\leq -w_1 + \frac{k(t)}{2} \left( \frac{-dA}{dt} \right). \end{aligned}$$

In turn, for  $t \geq t_0$  this yields the inequality

$$\begin{aligned} w_1(t, x(t), y(t)) &\leq e^{-(t-t_0)} \left[ w_1(t_0) + \int_{t_0}^t e^{s-t_0} k(s)(-A'(s)) ds \right] \\ &\leq e^{-(t-t_0)} [w_1(t_0) + A(t_0)k(t_0)] + e^{-(t-t_0)} \int_{t_0}^t A(s)(e^{s-t_0} k(s))' ds. \end{aligned} \quad (4.13)$$

The first term on the right hand side of (4.13) approaches zero exponentially fast. Concerning the second, we have

$$e^{-(t-t_0)} \int_{t_0}^t A(s) e^{s-t_0} (k(s) + k'(s)) ds \leq \int_{t_0}^t e^{-(t-s)} 2A_0 s^{4\theta-1} \left( \theta \ln s + \frac{\theta}{s} \right) ds.$$

This also approaches zero as  $t \rightarrow \infty$ . Repeating the same computations for all components, we conclude that for some time  $t_1$  sufficiently large there holds

$$w_i(t_1, x, y) < \frac{1}{2} \quad \text{for all } x, y \in [-1, 1]. \quad (4.14)$$

By continuity, the inequalities in (4.14) remain valid for all  $x, y$  in a slightly larger square, say  $[-1 - \epsilon, 1 + \epsilon]$ . For  $t \geq t_1$  we now define

$$M(t) \doteq \max \left\{ w_i(t, x, y); \quad i = 1, 2, 3, 4, \quad x, y \in [-1 - e^{t-t_1}\epsilon, 1 + e^{t-t_1}\epsilon] \right\}.$$

From the equations (3.2) and (4.14) it now follows

$$\frac{d}{dt} M(t) \leq -M(t) + M^2(t) \leq \frac{M(t)}{2}, \quad M(t_1) \leq \frac{1}{2}.$$

$$M(\tau) \leq \left[ 1 + e^{\tau-t_1} \left( \frac{1}{M_1} - 1 \right) \right]^{-1} \leq e^{-\tau} \cdot e^{t_1} \quad \text{for all } \tau \geq t_1.$$

Returning to the original variables  $u_i = e^\tau w_i$ , this yields

$$u_i \leq e^{t_1}$$

in a whole neighborhood of the point  $P^* = (t^*, x^*)$ . Hence  $P^*$  is not a blow-up point.  $\square$

## 5 - A tentative blow-up scenario

For a solution of the rescaled equation (3.1), the total mass

$$m(t) \doteq \int_{-1}^1 \int_{-1}^1 \sum_{i=1}^4 w_i(t, x, y) dx dy$$

may well become unbounded as  $t \rightarrow \infty$ . On the other hand, the one-dimensional integrals along horizontal or vertical segments decrease monotonically. Namely, if  $t \mapsto y(t)$  satisfies  $\dot{y} = y + 1$ , then

$$\frac{d}{dt} \int_{-1}^1 [w_1(t, x, y(t)) + w_4(t, x, y(t))] dx \leq 0.$$

Similarly, if  $\dot{x} = x - 1$ , then

$$\frac{d}{dt} \int_{-1}^1 [w_3(t, x(t), y) + w_4(t, x(t), y)] dy \leq 0.$$

Analogous estimates hold for the sums  $w_1 + w_2$  and  $w_2 + w_3$ . Therefore, a bound on the initial data

$$w_i(0, x, y) \in [0, M] \quad \text{for all } x, y \in [-1, 1],$$

yields uniform integral bounds on the line integrals of all components:

$$\int_{-1}^1 w_i(t, x, y) dx \leq 4M, \quad \int_{-1}^1 w_i(t, x, y) dy \leq 4M. \quad (5.1)$$

If finite time blow-up is to occur, the mass which is initially distributed along each horizontal or vertical segment must concentrate itself within a very small region, thus forming a narrow packet of particles with increasingly high density. A possible scenario is illustrated in fig. 3. A packet of 1-particles is initially located at  $P_1$ . In order to contribute to blow-up, this packet must remain within the unit square  $Q$ . At  $P_2$  these 1-particles interact with 3-particles and produce a packet of 4-particles. In turn, at  $P_3$  these interact with 2-particles and produce again a packet of 1-particles. After repeated interactions, the packet of alternatively 1- and 4-particles eventually enters within the smaller square  $Q'$ . After this time, it interacts with a packet of 2-particles at  $P_5$  (transforming it into a packet of 1-particles) and eventually exits from the domain  $Q$ .

To help intuition, it is convenient to describe a packet as being “young” until it enters the smaller square  $Q'$ , and “old” afterwards. To maintain a young packet inside  $Q$ , one needs the presence of old packets interacting with it near the points  $P_2, P_3, P_4 \dots$ . On the other hand, after it enters  $Q'$ , our packet can in turn be used to hit another young packet, say at  $P_5$ , and preventing it from leaving the domain  $Q$ .

As  $t \rightarrow \infty$ , the density of the packets must approach infinity. One thus expects that most of the mass will be concentrated along a finite number of one-dimensional curves. Say, the packet of alternatively 1- and 4-particles should be located along a moving curve  $\gamma_{14}(t, \theta)$ , where  $\theta$  is a parameter along the curve. The time evolution of such a curve is of course governed by the equations

$$\frac{\partial}{\partial t} \gamma_{14} = c_1 \quad \text{or} \quad \frac{\partial}{\partial t} \gamma_{14} = c_4$$

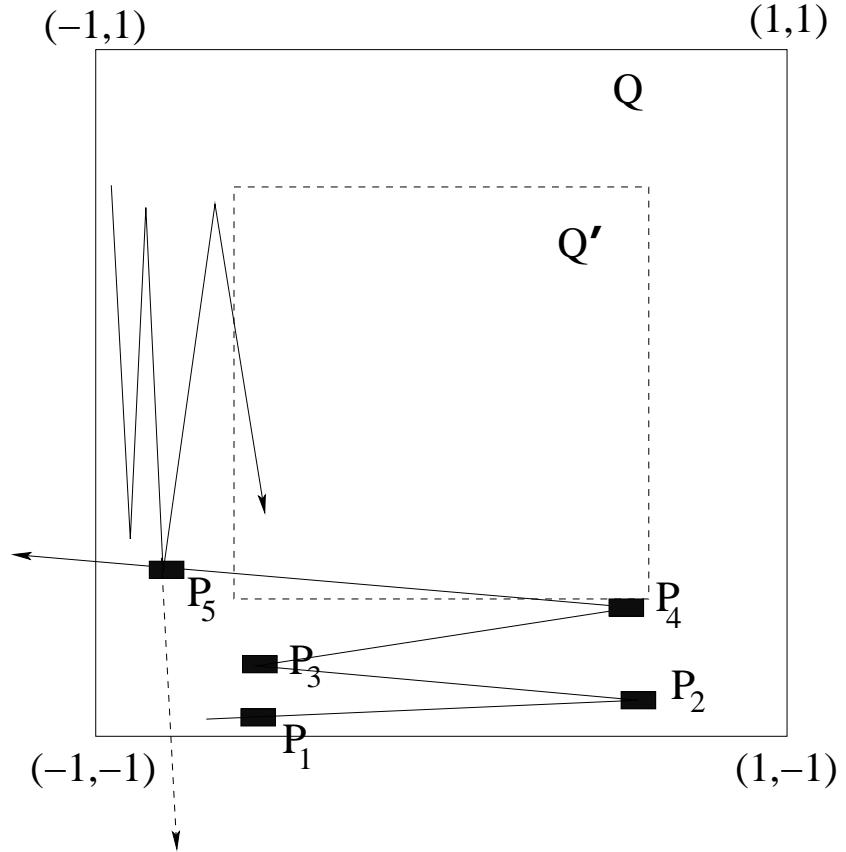


figure 3

depending on whether  $\gamma_{14}(t, \theta)$  consists of 1- or 4-particles. The presence of interactions impose highly nonlinear constraints on these curves. For example, the interaction occurring in  $P_5$  at time  $t$  implies the crossing of the two curves  $\gamma_{14}$  and  $\gamma_{12}$ , namely

$$\gamma_{14}(t, \theta) = \gamma_{12}(t, \tilde{\theta}) = P_5$$

for some parameter values  $\theta, \tilde{\theta}$ . The complicated geometry of these curves resulting from the above constraints has not been analyzed.

## References

- [1] J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Wiley, 1984.
- [2] J. M. Bony, Solutions globales bornées pour les modèles discrets de l'équation de Boltzmann en dimension 1 d'espace, *Actes Journées E.D.P. St. Jean de Monts* (1987).
- [3] J. M. Bony, Existence globale à donnée de Cauchy petites pour les modèles discrets de l'équation de Boltzmann, *Comm. Part. Diff. Equat.* **16** (1991), 533-545.

- [4] V. A. Galaktionov and J. L. Vazquez, The problem of blow-up in nonlinear parabolic equations, *Discr. Cont. Dyn. Syst.* **8** (2002), 399-433.
- [5] Y. Giga and R. Kohn, Characterizing blow-up using similarity variables, *Indiana Univ. Math. J.* **36** (1987), 1-40.
- [6] S. Y. Ha and A. Tzavaras, Lyapunov functionals and  $L^1$  stability for discrete velocity Boltzmann equations, *Comm. Math. Phys.* **239** (2003), 65-92.
- [7] R. Illner, Examples of non-bounded solutions in discrete kinetic theory, *J. Mécanique Th. Appl.* **5** (1986), 561-571.
- [8] R. Illner and T. Platkowski, Discrete velocity models of the Boltzmann equation. A survey on the mathematical aspects of the theory, *SIAM Review* **30** (1988), 213-255.
- [9] H. K. Jenssen, Blowup for systems of conservation laws, *SIAM J. Math. Anal.* **31** (2000), 894-908.
- [10] L. Tartar, Some existence theorem for semilinear hyperbolic systems in one space variable, *Technical Summary Report, Univ. Wisconsin* (1980).